UNIQUENESS OF SIGNATURE FOR SIMPLE CURVES

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Abstract. We propose a topological approach to the problem of determining a curve from its iterated integrals. In particular, we prove that a family of terms in the signature series of a two dimensional closed curve with finite p variation, $1 \leq p < 2$, are in fact moments of its winding number. This relation allows us to prove that the signature series of a class of simple non-smooth curves uniquely determine the curves. This implies that outside a Chordal SLE_κ null set, where $0 < \kappa \leq 4$, the signature series of curves uniquely determine the curves. Our calculations also enable us to express the Fourier transform of the n-point functions of SLE curves in terms of the expected signature of SLE curves. Although the techniques used in this article are deterministic, the results provide a platform for studying SLE curves through the signatures of its sample paths.

Keywords: Rough path theory; Uniqueness of signature problem; SLE curves.

1. Introduction

The signature of a path is a formal series of its iterated integrals. In [6], K.T. Chen observed that the map that sends a path to its signature forms a homomorphism from the concatenation algebra to the tensor algebra and used it to study the cohomology of loop spaces. Recent interests in the study of signature has been sparked by its role in the rough path theory. In particular, it was shown by Hambly and Lyons in [10] that for ODEs driven by paths with bounded total variations, the signature is a fundamental representation of the effect of the driving signal on the solution.

This article has two purposes:

1. To determine the winding number of a curve from its signature.

2. To prove, using a relation obtained from answering 1., that the signature of sufficiently regular planar simple curves uniquely determine the curves.

The first question was originally considered as far back as 1936, in a paper by Rado[19], who observed that the second term of the signature series of a smooth path is equal to the integral of its winding number around (x, y), considered as a function of (x, y). In [28], Yam considered the same problem as ours, but used a different approach. He started with the formula

$$\text{Winding number around } z = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\mathrm{d}w.$$

and smoothened the kernel $w \to \frac{1}{w-z}$ around the singularity at $w = z$. He then expanded $\frac{1}{w-z}$ into a power series of w and used the fact that the line integrals along γ of polynomials in w can be expressed in terms of the signature of γ.

Here we take a different approach and obtained a formula for the Fourier transform of the winding number, which appears to be simpler than the formula for the winding number itself. A classical result about iterated integrals, first proved by Chen [7], states that the logarithm of the signature of any path is a Lie series. The first result of this article states that the coefficient of some Lyndon basis elements in the log signature series are in fact moments of the winding number. In what follows, we will use some basic notions in free Lie algebra, which we shall recall in section 3. Throughout this article, we will use π_N to denote the projection of $T\left(\left(\mathbb{R}^d\right)\right)$ to $T^N\left(\mathbb{R}^d\right)$ (see section 2.1) and $S\left(\gamma\right)_{0,1}$ to denote the full signature of γ.

Theorem 1. *Let $1 \leq p < 2$. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with finite p variation. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the standard basis of $\mathbb{R}^2$. Define an order on $\{\mathbf{e}_1, \mathbf{e}_2\}$ by $\mathbf{e}_1 < \mathbf{e}_2$. Then*

1. For each $(n,k) \in \mathbb{N} \times \mathbb{N}$, $\mathbf{e}_1^{\otimes n} \otimes \mathbf{e}_2^{\otimes k}$ is a Lyndon word in the free Lie algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ with respect to the tensor product.

2. For each $n,k \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$, let $\mathcal{P}_{\mathbf{e}_1^{\otimes(n+1)} \otimes \mathbf{e}_2^{\otimes(k+1)}}$ be the Lyndon element corresponding to the Lyndon word $\{\mathbf{e}_1, \mathbf{e}_2\}$. Then, for all $n,k \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$, $N \geq n+k+2$, the coefficient of $\mathcal{P}_{\mathbf{e}_1^{\otimes n+1} \otimes \mathbf{e}_2^{\otimes k+1}}$ in the Lyndon basis expansion of $\pi_N(\log S(\gamma))$ is

$$(-1)^k \int_{\mathbb{R}^2} \frac{x^n y^k}{n!k!} \eta\left(\gamma - \gamma_0, (x,y)\right) \mathrm{d}x\mathrm{d}y. \tag{1.1}$$

where $\eta\left(\gamma - \gamma_0, (x,y)\right)$ is the winding number of the curve $\gamma - \gamma_0$ around the points $x\mathbf{e}_1 + y\mathbf{e}_2$.

As the winding number of a path does not contain information about the order at which it passes through points, whereas signature does, we cannot expect that the signature of a path can be expressed in terms of just winding numbers. In particular, let a and b be two closed curves in $\mathbb{R}^2$, both starting at 0 and let $\star$ denote the concatenation operation between two paths. Then $a \star b$ and $b \star a$ have the same winding number around any point, but in general do not have the same signature. Nevertheless, it is natural to ask how many terms in the signature series of a path can be represented in terms of its winding numbers. The answer is that the first four terms of a closed curve's signature can be expressed in terms of its winding number.

Corollary 2. *Let $1 \leq p < 2$. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with finite p variation. The first four terms of $\log\left(S\left(\gamma\right)_{0,1}\right)$ can be expressed in terms of the function $(x,y) \to \eta\left(\gamma - \gamma_0, (x,y)\right)$ alone.*

At the end of section three, we will prove that the number "four" is sharp. In other words, there are two paths $\gamma, \tilde{\gamma}$ which has the same winding number around every point, but the fifth terms of the signature of γ and $\tilde{\gamma}$ differs. The reason is that all Lynodn words of degree at most 4 generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ are of the form $\mathbf{e}_1^{\otimes n} \otimes \mathbf{e}_2^{\otimes k}$. On the other hand, there is a Lyndon word of degree 5 which is not of the form $\mathbf{e}_1^{\otimes n} \otimes \mathbf{e}_2^{\otimes k}$, namely, $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^{\otimes 2}$. This corresponds to the difficulty in expressing the iterated integral

$$\int_{0<s<t<1} \left[[\gamma_s, \mathrm{d}\gamma_s], [\gamma_t, [\gamma_t, \mathrm{d}\gamma_t]] \right]$$

in terms of the moments of the winding number of γ.

Uniqueness of signature

If we consider the signature as a representation of paths, then an interesting question is whether this representation is faithful. This was first considered by Chen himself [8], who proved that irreducible, piecewise regular continuous paths have the same signature if and only if they are equal up to a translation and a reparametrisation. His result was generalised with a new, quantitative approach by Hambly and Lyons in [10] who showed that two paths γ and $\tilde{\gamma}$ with finite total variations have the same signature if and only if γ can be expressed as the concatenation of $\tilde{\gamma}$ with a "tree-like" path σ.

Theorem 3. *Let $1 \leq p < 2$. Let γ, $\tilde{\gamma}$ be simple curves with finite p variation. Then $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$ if and only if γ and $\tilde{\gamma}$ are equal up to a translation and a reparametrisation.*

In the case of $p = 1$, we already know from the result of Hambly and Lyons that the simple curves can be recovered from the signature (modulo translation and reparametrisation) since simple curves have no tree-like parts. An interesting, but difficult extension is to prove that if the signatures of two curves with finite $p > 1$ -variations are equal, then the paths are equal up to the tree-like path equivalence. The restriction $1 \leq p < 2$ gives us the existence of signature for free, thanks to Young's integration theory.

Theorem 3 only applies to paths with finite p-variations, where $p < 2$. In particular, our results can only be applied to study stochastic processes whose sample paths are almost surely smoother than the Brownian motion sample paths. One example of such processes is the Chordal SLE_κ measure. The SLE measures were born from the study of lattice models which have conformally invariant scaling limit. There are a number of other lattice models whose scaling limit have been proved to be an SLE curve under some boundary conditions, such as the loop erased random walk ($\kappa = 2$, [12]), the Ising model ($\kappa = 3$, [5]), the level lines of Gaussian Free Field ($\kappa = 4$, [23]), percolation on the triangular lattice ($\kappa = 6$, [4] and [25]), and the Peano curve of the uniform spanning tree ($\kappa = 8$, [12]).

The path regularity and, in particular, the roughness of SLE curves, in relation to the speed κ of the driving Brownian motion, is an extremely interesting topic. It is intuitively clear that the SLE curves becomes rougher as the speed of the driving Brownian motion increases. In [11], the optimal Hölder exponent for SLE curves under the capacity parametrisation was proved to be

$$\min\left(\frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}\right).$$

In [2], V. Beffara proved that the almost Hausdorff dimension of SLE curves is $\min\left(1 + \frac{\kappa}{8}, 2\right)$. Therefore, the optimal Hölder exponent cannot exceed $\frac{1}{1+\frac{\kappa}{8}}$. B. Werness[27] proved that for $0 < \kappa \leq 4$, almost surely, the SLE curve in $\mathbb{D}$ has finite p variation for any $p > 1 + \frac{\kappa}{8}$. In another words, the roughness of an SLE curve grows linearly with the speed of the driving Brownian motion. It is strongly believed that this remains true for $4 < \kappa < 8$. However, to the best of our knowledge, this problem remains open.

In [27], B. Werness used his regularity result to define the signatures of SLE curves using Young's integral. He is also the first to realise that the Green's theorem can be used to compute some terms in the signature of a simple curve. He used it to prove the $n = 2, k = 1$ case of Lemma 19 for simple closed curves and to compute the first three gradings of the expected signature of SLE curve. Our work is inspired by and in fact generalises Werness's calculation. Later in Theorem 5, we shall show that our generalisation allows us to obtain the fourth term in the expected signature of SLE_κ curves. Werness method will not work to calculate fifth or later terms in the expected signature of SLE curves. This is because the fifth or later terms are not completely determined by the path's winding number.

In the study of SLE curves we often do not care about the curves' parametrisations and in some cases, it may be convenient to study the curves' signature instead. In order to do so, one must prove that there is a $1-1$ correspondence between curves and their signatures, outside a null set. Such injectiveness was proved for Brownian motion by Le Jan and Qian in [13] and for general diffusion processes by Geng and Qian. Both results rely on the Strong Markov property. Although the Chordal SLE_κ measure is not Markov, the inversion problem can be tackled for $\kappa \leq 4$ since the Chordal SLE_κ measure is supported on simple curves. The Chordal SLE_κ measure in a domain D is defined as the pull-back of the Chordal SLE_κ measure in $\mathbb{H}$ via a conformal map. Although the Chordal SLE_κ measure in $\mathbb{H}$ is parametrised on $[0, \infty)$, we know from [21] that the Chordal SLE_κ measure in $\mathbb{H}$ is supported on curves tending to infinity as time tends to infinity. This allows us to reparametised SLE_κ curves in a bounded Jordan domain D so that it is defined on $[0, \infty]$, or $[0, 1]$, by continuous extension. It follows from Theorem 3 that:

Theorem 4. *Let D be a bounded Dini-smooth Jordan domain and let a, b be two distinct boundary points of D. Let $\mathbb{P}^{a,b}_{\kappa,D}$ be the Chordal SLE_κ measure in D with marked points a and b. Then there exists a set of curves A, such that $\mathbb{P}^{a,b}_{\kappa,D}(A^c) = 0$ for all $0 < \kappa \leq 4$ and if $\gamma, \tilde{\gamma} \in A$ and $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$, then γ and $\tilde{\gamma}$ are equal up to a reparametrisation.*

The Dini-smooth condition was introduced to ensure the existence of a Lipschitz conformal map from $\mathbb{D}$ to D. See [19] for a proof of this result and the definition of Dini-smooth. This ensures that the SLE_κ curves in D has the same regularity as the SLE_κ curves in $\mathbb{D}$.

The expected signature can be considered as the "Laplace's transform" of a stochastic process and has first been studied in [8]. The sequence of n-point functions of the Chordal SLE measure was first studied by O. Schramm. Using a generalised Green's theorem for non-smooth curves, we may prove the following relationship between the expected signature and the sequence of n-point functions.

Theorem 5. *Let $0 \leq \kappa \leq 4$. Let D be a bounded Dini-smooth Jordan domain and $a, b \in \partial D$. Let $\mathbb{P}^{a,b}_{\kappa,D}$ be the Chordal SLE_κ measure in D with marked points a and b. For each curve γ, let $\Phi(\gamma)$ denote the concatenation of γ with the positively oriented arc of ∂D from b to a. For each $N \in \mathbb{N}$, let Γ_N denotes the n-point function associated with $\mathbb{P}^{a,b}_{\kappa,D}$, then for all $N \geq 1$ and $\lambda_i, \mu_i \in \mathbb{R}$ for $i = 1, \ldots N$,*

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} e^{\sum_{i=1}^{N}\lambda_i x_i+\mu_i y_i}\Gamma_N\left(\left(x_1,y_1\right),\ldots,\left(x_N,y_N\right)\right)\mathrm{d}x_1\cdots\mathrm{d}y_N \\
=&\sum_{n_1,\ldots,n_N,k_1\ldots k_N\geq 0}\Pi_{i=1}^{N}\left(\lambda_i\right)^{n_i}\left(-\mu_i\right)^{k_i}\mathbf{e}_1^{*\otimes(n_1+1)}\otimes\mathbf{e}_2^{*\otimes(k_1+1)}\sqcup\ldots \\
&\ldots\sqcup\mathbf{e}_1^{*\otimes(n_N+1)}\otimes\mathbf{e}_2^{*\otimes(k_N+1)}\left(\mathbb{E}_{\kappa,D}^{a,b}\left[S\left(\Phi\left(\cdot\right)\right)_{0,1}\right]\right).
\end{aligned}$$

where $\mathbf{e}_i^*$ *is the dual basis corresponding the standard basis of* $\mathbb{R}^2$ *(see section 2.1) and* $\sqcup$ *denotes the shuffle product (see Proposition 7).*

The plan for the rest of the article is as follows.
In section 2, we recall the basic results about the signature and winding number.
In section 3, we prove Theorem 1 and Corollary 2.
In section 4, we prove Theorem 3.
In section 5, we prove Theorem 4.
In section 6, we prove Theorem 5.

2. Preliminaries

2.1. **Basic notations.** Let $T\left(\left(\mathbb{R}^d\right)\right)$ be the set of sequences

$$(a_0,a_1,a_2,\ldots)$$

where $a_i\in\left(\mathbb{R}^d\right)^{\otimes i}$. equipped with the addition and multiplication operation $+$ and $\otimes$. The binary operations $+$ and $\otimes$ are defined so that for all $\mathbf{a},\mathbf{b}\in T\left(\left(\mathbb{R}^d\right)\right)$, if $\pi^{(i)}$ denote the projection of a sequence onto its i th term, then

$$\pi^{(n)}\left(\mathbf{a}+\mathbf{b}\right):=\pi^{(n)}\left(\mathbf{a}\right)+\pi^{(n)}\left(\mathbf{b}\right) \tag{2.1}$$

and

$$\pi^{(n)}\left(\mathbf{a}\otimes\mathbf{b}\right):=\sum_{i=0}^{n}\pi_i\left(\mathbf{a}\right)\otimes\pi_i\left(\mathbf{b}\right). \tag{2.2}$$

$T\left(\left(\mathbb{R}^d\right)\right)$ is called the formal series of tensors of $\mathbb{R}^d$.

Let $T^k\left(\mathbb{R}^d\right)$ denote the set of all finite k-sequences

$$(a_0,\ldots,a_k)$$

where $a_i\in\left(\mathbb{R}^d\right)^{\otimes i}$. The addition and mulitplication operation, $+$ and $\otimes$, on $T^k\left(\mathbb{R}^d\right)$ are defined by (2.1) and (2.2) for $n=0,1,\ldots,k$. We will use π_k to denote the projection map from $T\left(\mathbb{R}^d\right)$ to $T^k\left(\mathbb{R}^d\right)$.

For each $f_1,\ldots,f_k\in\left(\mathbb{R}^d\right)^*$ define $f_1\otimes\ldots\otimes f_k$ on $\left(\mathbb{R}^d\right)^{\otimes k}$ by extending linearly the relation

$$f_1\otimes\ldots\otimes f_k\left(v_1\otimes\ldots\otimes v_k\right):=f_1\left(v_1\right)\ldots f_n\left(v_k\right).$$

We may extend the map $f_1\otimes\ldots\otimes f_k$ to a functional on $T\left(\left(\mathbb{R}^d\right)\right)$ by defining for all $\mathbf{a}\in T\left(\left(\mathbb{R}^d\right)\right)$,

$$f_1\otimes\ldots\otimes f_k\left(\mathbf{a}\right):=f_1\otimes\ldots\otimes f_k\left(\pi^{(k)}\left(\mathbf{a}\right)\right).$$

2.2. **Signature.** Let $p>1$ and let $\mathcal{V}^p\left([0,1],\mathbb{R}^d\right)$ denote the set of all continuous functions $\gamma:[0,1]\to\mathbb{R}^d$ such that

$$\|\gamma\|^p_{\mathcal{V}^p([0,1],\mathbb{R}^d)} := \sup_{\mathcal{P}} \sum_k \left|\gamma_{t_{k+1}} - \gamma_{t_k}\right|^p < \infty. \tag{2.3}$$

where the supremum is taken over all finite partitions $\mathcal{P} := (t_0, t_1, .., t_{n-1}, t_n)$, where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = 1$.

The elements of $\mathcal{V}^p\left([0,1],\mathbb{R}^d\right)$ will be called curves with finite p-variation. This class of paths with finite p-variation is narrower than the one used by Young [29] because we restrict our considerations to *continuous* paths.

Note that $\|\cdot\|_{\mathcal{V}^p([0,1],\mathbb{R}^d)}$ defines a semi-norm on $\mathcal{V}^p\left([0,1],\mathbb{R}^d\right)$.

Definition 6. Let $1\le p<2$. Let $\gamma\in\mathcal{V}^p\left(\mathbb{R}^d\right)$and let $\triangle_n(s,t) := \{(t_1,\dots,t_n) : s<t_1<\dots<t_n<t\}$. The *lift* of γ is a function $S(\gamma)_{\cdot,\cdot} : \{(s,t) : 0\le s\le t\}\to T\left(\left(\mathbb{R}^d\right)\right)$ defined by

$$S(\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty} \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1}\otimes\ldots\otimes\mathrm{d}\gamma_{t_n} \tag{2.4}$$

where the integrals are taken in the sense of Young [29].

The signature of a path $\gamma\in\mathcal{V}^p\left(\mathbb{R}^d\right)$ on $[0,1]$ is defined to be $S(\gamma)_{0,1}$.

We shall use the following properties of signature, whose proofs can be found in [14] or .

1. (Invariance under reparametrisation)For any $t\in[0,\infty)$, $S(\gamma)_{0,t}$ is invariant under any reparametrisation of γ on $[0,t]$.

2. (Inverse) $S(\gamma)_{0,1}\otimes S(\overleftarrow{\gamma})_{0,1} = \mathbf{1}$, where $\overleftarrow{\gamma}(t) := \gamma(1-t)$ is the reversal of γ and $\mathbf{1}$ is the identity element in $T\left(\mathbb{R}^d\right)$.

3. (Chen's Identity)$S(\gamma)_{s,u}\otimes S(\gamma)_{u,t} = S(\gamma)_{0,t}$ for any $0\le s<u<t\le 1$

4. (Scaling and translation)Let $\lambda\in\mathbb{R}^d$, $\mu\in\mathbb{R}$, then

$$S(\lambda+\mu\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty}\mu^n\int_{\triangle_n(s,t)}\mathrm{d}\gamma(t_1)\otimes\ldots\otimes\mathrm{d}\gamma(t_n)$$

5.(Lie Series) $\log S(\gamma)_{0,1}$ is a Lie series.

6.(Shuffle product formula) We define a *(r,s)-shuffle* to be a permutation of $\{1,2,...,r+s\}$ such that $\sigma(1)<\sigma(2)<..<\sigma(r)$ and $\sigma(r+1)<...<\sigma(r+s)$.

Proposition 7. *([14],Theorem 2.15) Let $1\le p<2$ and $\gamma\in\mathcal{V}^p\left([0,1],\mathbb{R}^d\right)$, then*

$$\begin{aligned}&\mathbf{e}^*_{k_1}\otimes\ldots\otimes\mathbf{e}^*_{k_r}\left(S(\gamma)_{0,1}\right)\mathbf{e}^*_{k_{r+1}}\otimes\ldots\otimes\mathbf{e}^*_{k_{r+s}}\left(S(\gamma)_{0,1}\right)\\ =&\sum_{(r,s)-shuffles\ \sigma}\mathbf{e}^*_{k_{\sigma^{-1}(1)}}\otimes\ldots\otimes\mathbf{e}^*_{k_{\sigma^{-1}(r+s)}}\left(S(\gamma)_{0,1}\right).\end{aligned}$$

where $\cdot$ is the multiplication operation in $\mathbb{R}$.

The sum

$$\sum_{(r,s)-shuffles\ \sigma}\mathbf{e}^*_{k_{\sigma^{-1}(1)}}\otimes\ldots\otimes\mathbf{e}^*_{k_{\sigma^{-1}(r+s)}}$$

is denoted by $\mathbf{e}^*_{k_1}\otimes\ldots\otimes\mathbf{e}^*_{k_r}\sqcup\mathbf{e}^*_{k_{r+1}}\otimes\ldots\otimes\mathbf{e}^*_{k_{r+s}}$.

We shall need a few approximation theorems relating the p-variation of a path with its piecewise linear interpolations. For a continuous function γ and a partition

$\mathcal{P} := t_0 = 0 < t_1 < .. < t_n = 1$, the piecewise linear interpolation of γ with respect to $\mathcal{P}$ is defined as the following function on $[0,T]$:

$$\gamma_t^{\mathcal{P}} := \gamma_{t_i} + \left(\frac{\gamma_{t_{i+1}} - \gamma_{t_i}}{t_{i+1} - t_i}\right)(t - t_i) \text{ for } t \in [t_i, t_{i+1}]$$

Then the following approximation theorem holds:

Lemma 8. *(Lemma 1.12 and Proposition 1.14,* [14]*)Let p and q be such that $1 \leq p < q$. Let $\gamma \in \mathcal{V}^p\left([0,1], \mathbb{R}^d\right)$. Then for all finite partitions $\mathcal{P}$,*

$$\left\|\gamma^{\mathcal{P}}\right\|_{\mathcal{V}^p([0,1],\mathbb{R}^d)} \leq \|\gamma\|_{\mathcal{V}^p([0,1],\mathbb{R}^d)}$$

Furthermore for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all partitions $\mathcal{P}$ of $[0,1]$ satisfying $\|\mathcal{P}\| < \delta$ we have

$$\begin{aligned} \left\|\gamma - \gamma^{\mathcal{P}}\right\|_{\mathcal{V}^q([0,1],\mathbb{R}^d)} &< \varepsilon, \text{ and} \\ \sup_{t\in[0,1]} \left\|\gamma_t - \gamma_t^{\mathcal{P}}\right\| &< \varepsilon. \end{aligned}$$

The following lemma is extremely useful in proving the properties of Young's integral.

Lemma 9. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous curve with finite p-variation, where $p < 2$. Let $\mathcal{P}_m$ be a sequence of partitions such that $\mathcal{P}_m$ contains both 0 and 1 for all m and $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$. For any $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$,*

$$\mathbf{e}_{i_1}^* \otimes \ldots \otimes \mathbf{e}_{i_n}^* \left[S(\gamma)_{0,1}\right] = \lim_{m\to\infty} \mathbf{e}_{i_1}^* \otimes \ldots \otimes \mathbf{e}_{i_n}^* \left[S\left(\gamma_s^{\mathcal{P}_m}\right)_{0,1}\right]. \tag{2.5}$$

Proof. See Corollary 2.11 in [14]. □

2.3. **Winding number.** In this section, we shall recall the definition of winding number and a few key basic facts that we shall use.

Definition 10. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous function. Then
1.γ is a closed curve if $\gamma_0 = \gamma_1$.
2.γ is a simple closed curve if $\gamma_s = \gamma_t$ implies either $s = t$ or $\{s,t\} = \{0,1\}$.
3.γ is a simple curve if $\gamma_s = \gamma_t$ implies $s = t$.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous function. Let $z \in \mathbb{R}^2 \backslash \gamma[0,1]$. Then

$$g_z^\gamma(s) := \frac{\gamma_s - z}{\|\gamma_s - z\|}$$

defines a function $[0,1] \to \mathbb{S}^1$.

Let $p : \mathbb{R} \to \mathbb{S}^1$, $p(x) = e^{ix}$ be a covering map for $\mathbb{S}^1$. Then there exists a continuous lift $\tilde{g}_z^\gamma : [0,1] \to \mathbb{R}$ such that $p \circ \tilde{g}_z^\gamma = g_z^\gamma$. The winding number of γ will be defined in terms of $\tilde{g}_s(z)$ by the following lemma:

Lemma 11. *(*[18]*, Chapter 3 Lemma 1 and 2)Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve, and $z \in \gamma[0,1]$. Then the number*

$$\eta(\gamma, z) := \frac{1}{2\pi}\left(\tilde{g}_z^\gamma(1) - \tilde{g}_z^\gamma(0)\right) \tag{2.6}$$

depends only on γ and z but not on the lift $\tilde{g}_z^\gamma$. Moreover, $\eta(\gamma, z)$ is an integer and is called the winding number of γ around the point z.

Remark 12. We may define the winding number for any $\gamma : [a, b] \to \mathbb{R}^2$ by simply replacing 0 by a, 1 by b in the above definition.

The following theorem, which we shall need, is intuitively clear but is highly non-trivial:

Theorem 13. *([17], p404)Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a simple closed curve. Let Int(γ) and Ext(γ) be its interior and exterior respectively. Then $\eta(\gamma, z) = 0$ for all $z \in$Ext(γ). Moreover, either $\eta(\gamma, z) = 1$ for all $z \in Int(\gamma)$ or $\eta(\gamma, z) = -1$ for all $z \in Int(\gamma)$. γ is called positively oriented if $\eta(\gamma, z) = 1$ and negatively oriented otherwise.*

A key tool in our proof of Proposition 1 is the following Green's theorem for paths with bounded total variations.

Theorem 14. *([19] and [1])Let $\gamma = \left(\gamma^{(1)}, \gamma^{(2)}\right) : [0, T] \to \mathbb{R}^2$ be a closed curve with bounded total variation. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives in both variables. Then*

$$\int_{\mathbb{R}^2} \left(\partial_x f(x, y) + \partial_y g(x, y)\right) \eta(\gamma, (x, y))\, \mathrm{d}x\mathrm{d}y = \int_\gamma f \mathrm{d}\gamma_s^{(2)} - g\mathrm{d}\gamma_s^{(1)}. \tag{2.7}$$

and

$$\|\eta(\gamma, \cdot)\|_{L^2} \le \frac{1}{\sqrt{4\pi}} \|\gamma\|_{\mathcal{V}^1([0,T],\mathbb{R}^2)} \tag{2.8}$$

where the equality in (2.8) holds if and only if there exists $(x, y) \in \mathbb{R}^2$, $n \in \mathbb{N}$ and $R > 0$ such that $\gamma_t = (x + R\cos 2\pi nt, x + R\sin 2\pi nt)$.

The $f(x, y) = x$, $g(x, y) = y$ case in (2.7) was proved in [19] and the proof for the general case is essentially the same. New, complete proofs for (2.7) were subsequently given by [26] and [28].

The second inequality is the well-known Banchoff-Pohl isoperimetric inequality[1].

3. Proof of Theorem 1

Before we give a proof of Theorem 1, we would like to first recall some elementary Lie algebra.

3.1. Lyondon basis.

Let $\mathcal{L}(\{\mathbf{e}_1, \mathbf{e}_2\})$ be the set of Lie series generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ through the tensor product $\otimes$ and let $\mathcal{L}_N(\{\mathbf{e}_1, \mathbf{e}_2\}) := \pi_N(\mathcal{L}(\{\mathbf{e}_1, \mathbf{e}_2\}))$. We shall recall the definition of the Lyndon basis, which we used in decompose $\pi_N\left(\log S(\gamma)_{0,1}\right)$ in Theorem 1.

From here onwards, a *word* will mean a non-empty monomial generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ through $\otimes$. We shall assign an lexigraphical order on the set of words by the following rule:

(1) $\mathbf{e}_1 < \mathbf{e}_2$.
(2) If $\mathbf{v} = \mathbf{u} \otimes \mathbf{x}$ for some word $\mathbf{x}$, then $\mathbf{u} < \mathbf{v}$.
(3) If $\mathbf{w} = \mathbf{u} \otimes \mathbf{e}_1 \otimes \mathbf{x}$ and $\mathbf{w}' = \mathbf{u} \otimes \mathbf{e}_2 \otimes \mathbf{x}'$ for words $\mathbf{u}, \mathbf{x}, \mathbf{x}'$, then $\mathbf{w} < \mathbf{w}'$.

We say a word $\mathbf{w}$ is Lyndon if either $\mathbf{w} = \mathbf{e}_1$ or $\mathbf{w} = \mathbf{e}_2$ or for all $\mathbf{u} \neq 1$,$\mathbf{v} \neq 1$ such that $\mathbf{u} \otimes \mathbf{v} = \mathbf{w}$, we have $\mathbf{w} < \mathbf{v}$. For each word $\mathbf{w}$, $\mathbf{w} \neq \mathbf{e}_1, \mathbf{e}_2$, if $\mathbf{v}$ is the smallest non-empty word such that $\mathbf{w} = \mathbf{u} \otimes \mathbf{v}$ for some non-empty word $\mathbf{u}$, then we say $\mathbf{w} = \mathbf{u} \otimes \mathbf{v}$ is the standard factorisation of a Lyndon word $\mathbf{w}$.

Example 15. The Lyndon words of degree less than or equal 4 generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ are

$$\mathbf{e}_1 < \mathbf{e}_1^{\otimes 3} \otimes \mathbf{e}_2 < \mathbf{e}_1^{\otimes 2} \otimes \mathbf{e}_2 < \mathbf{e}_1^{\otimes 2} \otimes \mathbf{e}_2^{\otimes 2} < \mathbf{e}_1 \otimes \mathbf{e}_2 < \mathbf{e}_1 \otimes \mathbf{e}_2^{\otimes 2} < \mathbf{e}_1 \otimes \mathbf{e}_2^{\otimes 3} < \mathbf{e}_2.$$

For each Lyndon word, we can associate a corresponding Lyndon element $\mathcal{P}_{\mathbf{w}}$ inductively by $\mathcal{P}_{\mathbf{e}_1} = \mathbf{e}_1,\ \mathcal{P}_{\mathbf{e}_2} = \mathbf{e}_2$ and $\mathcal{P}_{\mathbf{w}} = [\mathcal{P}_{\mathbf{u}}, \mathcal{P}_{\mathbf{v}}]$ if $\mathbf{w} = \mathbf{uv}$ is the standard factorisation. By Theorem 4.9 and Theorem 5.1 in [20], the set

$$\{\mathcal{P}_{\mathbf{w}} : \mathbf{w} \text{ is a Lyndon word}\}$$

forms a basis of $\mathcal{L}(\{\mathbf{e}_1, \mathbf{e}_2\})$.

We shall now state a few key properties of the Lyndon words which we shall use.

Lemma 16. *1. ([20], (5.1.2))Let $\mathbf{u} < \mathbf{v}$ be two Lyndon words. Then $\mathbf{u} \otimes \mathbf{v}$ is also a Lyndon word.*

2.([20], Theorem 5.1) Let $n \in \mathbb{N}$. Let $\mathbf{w}$ be a Lyndon word such that $\mathbf{w} = \mathbf{l_1} \dots \mathbf{l_n}$, where $\mathbf{l_1} \geq \mathbf{l_2} \geq \dots \geq \mathbf{l_n}$ are Lyndon words. Then $\mathcal{P}_{\mathbf{w}} = \mathbf{w} + h.o.t$ where $h.o.t$ is a linear combination over $\mathbb{Z}$ of words strictly greater than $\mathbf{w}$.

From which it follows easily that:

Corollary 17. *$\mathbf{e}_1^{\otimes n} \mathbf{e}_2^{\otimes k}$ is a Lyndon word for all $n > 0$ and $k > 0$.*

Proof. Iterative use of 1. in Lemma 16. □

3.2. **Proof of Theorem 1.** We first need a technical lemma which controls the L^q norm of the winding number.

Lemma 18. *Let $1 \leq p < 2$. Then for all $q < \frac{2}{p}$, there exists $C_{p,q} > 0$ such that for all paths $\gamma : [0,1] \to \mathbb{R}^2$ with finite p-variation,.*

$$\|\eta(\gamma, \cdot)\|_{L^q} \leq C_{p,q} \max\left(\|\gamma\|_p, \|\gamma\|_p^p\right).$$

Proof. First consider the case when γ has finite total variation. For such paths, we have the integral representation

$$\eta(\gamma, (x,y)) = \frac{1}{2\pi} \int_0^1 \frac{(x_s - x)\,\mathrm{d}y_s - (y_s - y)\,\mathrm{d}x_s}{(x_s - x)^2 + (y_s - y)^2}.$$

Let $f \in L^q(\mathbb{R}^2)$, where $q > \frac{2}{2-p}$. Consider the map $f \to \int_{\mathbb{R}^2} f(z)\,\eta(\gamma)(z)\,\mathrm{d}z$. By an interchange of integral, we have

$$\int_{\mathbb{R}^2} f(x,y)\,\eta(\gamma,(x,y))\,\mathrm{d}x\mathrm{d}y \quad = \quad \frac{1}{2\pi}\left(\mathbf{e}_2^* \otimes \mathbf{e}_1^* - \mathbf{e}_1^* \otimes \mathbf{e}_2^*\right) \int_0^1 \left(\int_{\mathbb{R}^2} \frac{\gamma_s - (x,y)}{|\gamma_s - (x,y)|^2} f(x,y)\,\mathrm{d}x\mathrm{d}y\right) \otimes \mathrm{d}\gamma_s.$$

The quasi-potential operator T defined by

$$T(f)(z) := \int_{\mathbb{R}^2} \frac{z - (x,y)}{|z - (x,y)|^2} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

is a bounded linear operator from $L^q(\mathbb{R}^2)$ to $\mathrm{Lip}\left(1 - \frac{2}{q}\right)$ (See Theorem 3.7.1 in [Mor66]).

Note that as $q > \frac{2}{2-p}$, $2 - \frac{2}{q} > p$. Therefore,

$$\left|\int_0^1 (Tf)(\gamma_s) \otimes \mathrm{d}\gamma_s\right| \quad \leq \quad C_{p,q} \|(Tf)(\gamma_\cdot)\|_{\mathrm{Lip}(2-\frac{2}{q})} \max\left(\|\gamma\|_p^p, \|\gamma\|_p\right).$$

Therefore, the map

$$f \to \int_{\mathbb{R}^2} f(x,y)\,\eta(\gamma,(x,y))\,\mathrm{d}x\mathrm{d}y$$

is a bounded linear functional on L^q and

$$\left|\int_{\mathbb{R}^2} f(x,y)\,\eta(\gamma,(x,y))\,\mathrm{d}x\mathrm{d}y\right| \le C_{p,q}\,\|f\|_q \max\left(\|\gamma\|_p^p, \|\gamma\|_p\right).$$

This means for all paths γ with bounded total variation, and all $q > \frac{2}{2-p}$, or $q' < \frac{2}{p}$,

$$\|\eta(\gamma,\cdot)\|_{L^{q'}} \le C_{p,q} \max\left(\|\gamma\|_p^p, \|\gamma\|_p\right)$$

where $C_{p,q}$ is a constant independent of γ.

Let γ now be a path with finite p -variation, where $p < 2$. Let $\mathcal{P}$ be any piecewise linear interpolation of γ. Then

$$\begin{aligned}\left\|\eta\left(\gamma^{\mathcal{P}},\cdot\right)\right\|_{L^{q'}} &\le C\max\left(\left\|\gamma^{\mathcal{P}}\right\|_p^p, \left\|\gamma^{\mathcal{P}}\right\|_p\right)\\ &\le C\max\left(\|\gamma\|_p^p, \|\gamma\|_p\right).\end{aligned}$$

Let $\mathcal{P}_n$ be a sequence of partitions such that $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$. Then by Fatou's Lemma,

$$\begin{aligned}\lim_{n\to\infty}\left\|\eta\left(\gamma^{\mathcal{P}_n}\right)(\cdot)\right\|_{L^{q'}} &\le \|\eta(\gamma)(\cdot)\|_{L^{q'}}\\ &\le C\max\left(\|\gamma\|_p^p, \|\gamma\|_p\right).\end{aligned}$$

□

A key idea in proving Theorem 1 lies in the fact that the coefficients of some Hall basis elements can be reduced to a single line integral, as illustrated by the following lemma.

Lemma 19. *Let* $1 \le p < 2$. *Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a continuous closed curve with finite p variation. Let* $\eta(\gamma,(x,y))$ *denote the winding number of* γ *around* $x\mathbf{e}_1 + y\mathbf{e}_2$.

Then for all $n, k \ge 0$,

$$\mathbf{e}_1^{*\otimes(n+1)} \otimes \mathbf{e}_2^{*\otimes(k+1)}\left(S(\gamma)_{0,1}\right) = \frac{(-1)^k}{n!k!}\int_{\mathbb{R}^2} x^n y^k \eta(\gamma-\gamma_0;(x,y))\,\mathrm{d}x\mathrm{d}y. \tag{3.1}$$

Proof. We first prove the lemma for paths with bounded total variation.

Let $\gamma^{(1)}$ and $\gamma^{(2)}$ be the first and second coordinate components of γ respectively.

Recall that for all $n, k \ge 0$,

$$\mathbf{e}_1^{*\otimes(n+1)} \otimes \mathbf{e}_2^{*\otimes(k+1)}\left(S(\gamma)_{0,1}\right) = \int_{\triangle_{n+k+2}(0,1)} \mathrm{d}\gamma_{s_1}^{(1)}\ldots\mathrm{d}\gamma_{s_{n+1}}^{(1)}\mathrm{d}\gamma_{s_{n+2}}^{(2)}\ldots\mathrm{d}\gamma_{s_{n+k+2}}^{(2)}.$$

The key idea here is to integrate with respect to $\gamma^{(1)}$s first and then integrate the $\gamma^{(2)}$s. For all $n,k\geq 0$,

$$\begin{aligned}
&\mathbf{e}_1^{*\otimes(n+1)}\otimes\mathbf{e}_2^{*\otimes(k+1)}\left(S\left(\gamma\right)_{0,1}\right)\\
=&\int\ldots\int_{0<t_1<..<t_{n+1}<s_1<\ldots<s_{k+1}<1}\mathrm{d}\gamma_{t_1}^{(1)}\ldots\mathrm{d}\gamma_{t_{n+1}}^{(1)}\mathrm{d}\gamma_{s_1}^{(2)}\ldots\mathrm{d}\gamma_{s_{k+1}}^{(2)}\\
=&\int_{0<s_1<\ldots<s_{k+1}<1}\frac{1}{n!}\left(\gamma_{s_1}^{(1)}-\gamma_0^{(1)}\right)^{n+1}\mathrm{d}\gamma_{s_1}^{(2)}\ldots\mathrm{d}\gamma_{s_{k+1}}^{(2)}\\
=&\int_0^1\int_{s_1}^1\ldots\int_{s_{k-1}}^1\int_{s_k}^1\frac{1}{n!}\left(\gamma_{s_1}^{(1)}-\gamma_0^{(1)}\right)^{n+1}\mathrm{d}\gamma_{s_{k+1}}^{(2)}\ldots\mathrm{d}\gamma_{s_1}^{(2)}\text{ by Fubini's theorem}\\
=&\frac{1}{(n+1)!}\frac{1}{k!}\int_0^1\left(\gamma_{s_1}^{(1)}-\gamma_0^{(1)}\right)^{n+1}\left(\gamma_1^{(2)}-\gamma_{s_1}^{(2)}\right)^k\mathrm{d}\gamma_{s_1}^{(2)}\\
=&\frac{1}{n!}\frac{1}{k!}\int_{\mathbb{R}^2}\left(x-\gamma_0^{(1)}\right)^n\left(\gamma_1^{(2)}-y\right)^k\eta\left(\gamma;(x,y)\right)\mathrm{d}x\mathrm{d}y\text{ by (2.7)}\\
=&\frac{(-1)^k}{n!k!}\int_{\mathbb{R}^2}x^ny^k\eta\left(\gamma-\gamma_0;(x,y)\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

where in the last two steps we have used the fact that γ is a closed curve.

Now for γ with finite p variation, for each $N\in\mathbb{N}$, let $\mathcal{P}_N$ denote a sequence of partitions of $[0,1]$ such that $\|\mathcal{P}_N\|\to 0$ as $N\to\infty$. Then by what we just proved, (3.2)

$$\mathbf{e}_1^{*\otimes(n+1)}\otimes\mathbf{e}_2^{*\otimes(k+1)}\left(S\left(\gamma^{\mathcal{P}_n}\right)_{0,1}\right)=\frac{(-1)^k}{n!k!}\int_{\mathbb{R}^2}x^ny^k\eta\left(\gamma^{\mathcal{P}_N}-\gamma_0;(x,y)\right)\mathrm{d}x\mathrm{d}y.$$

We will now take limit as $N\to\infty$. The left hand side of (3.2) converges to $S\left(\gamma\right)_{0,1}$ by Corollary 2.17 in [14].

To show the right hand side of (3.2) converges to

$$\frac{(-1)^k}{n!k!}\int_{\mathbb{R}^2}x^ny^k\eta\left(\gamma-\gamma_0;(x,y)\right)\mathrm{d}x\mathrm{d}y$$

, note that by Lemma 18, if we take $1<q<\frac{2}{p}$,

$$\left\|\eta\left(\gamma^{\mathcal{P}_N},\cdot\right)\right\|_{L^q}\leq C_{p,q}\max\left(\left\|\gamma^{\mathcal{P}_N}\right\|_p,\left\|\gamma^{\mathcal{P}_N}\right\|_p^p\right)\leq C_{p,q}\max\left(\|\gamma\|_p,\|\gamma\|_p^p\right)$$

and the convergence follows from L^q convergence theorems. □

We will now give a proof of Theorem 1.

Proof. (of Theorem 1)Let $n,k\geq 0$ and $N\geq n+k+2$. If we equip the alphabet $\{\mathbf{e}_1,\mathbf{e}_2\}$ with the ordering $\mathbf{e}_1<\mathbf{e}_2^*$, then by Lemma 16, $\mathbf{e}_1{}^{\otimes n}\otimes\mathbf{e}_2{}^{\otimes k}$ is a Lyndon word as defined in section 3.1. Let $\mathcal{P}_{\mathbf{e}_1^{\otimes n}\otimes\mathbf{e}_2^{\otimes k}}$ denote the corresponding Lyndon element. By Lemma 19, it suffices to prove that for all $n,k\geq 0$ and $N\geq n+k+2$,

$$\mathcal{P}^*_{\mathbf{e}_1{}^{\otimes n+1}\otimes\mathbf{e}_2{}^{\otimes k+1}}\left(\log S_N\left(\gamma\right)_{0,1}\right)=\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(S_N\left(\gamma\right)_{0,1}\right).$$

We will first prove that for closed curve γ, for all $n\geq 0$,$k\geq 0$,

$$\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(\left(\log S\left(\gamma\right)_{0,1}\right)^{\otimes j}\right)=0\tag{3.3}$$

for $j\geq 2$.

First note that as γ is a closed curve

$$\mathbf{e}_1^*\left(\log S(\gamma)_{0,1}\right)=\mathbf{e}_2^*\left(\log S(\gamma)_{0,1}\right)=0. \tag{3.4}$$

If we denote the coefficient of a word w in a polynomial $\mathcal{P}$ by $(\mathcal{P},w)$, then for all $n,k\geq 0$,

$$\begin{aligned}&\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(\left(\log S(\gamma)_{0,1}\right)^{\otimes j}\right)\\ =&\sum_{w_1\ldots w_j=\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}}\left(\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right),w_1\right)\ldots\left(\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right),w_j\right).\end{aligned}$$

For each ordered collection of words $w_1,\ldots,w_j$ satisfying $w_1,\ldots,w_j=\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}$, then at least one of $w_1,\ldots,w_j$ will be of the form $\mathbf{e}_i^{\otimes l}$ where $i=1$ or 2 for some $l\geq 1$. Without loss of generality, assume this word is w_1. As $\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right)$ is a Lie polynomial and the first degree term of $\log S(\gamma)_{0,1}$is zero (see (3.4)),

$$\left(\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right),w_1\right)=0$$

which proves (3.3).

Therefore, for all $n,k\geq 0$,

$$\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(S(\gamma)_{0,1}\right)\quad=\quad\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(\log S(\gamma)_{0,1}\right).$$

We now equip the alphabet $\{\mathbf{e}_1,\mathbf{e}_2\}$ with the ordering $\mathbf{e}_1<\mathbf{e}_2$, so that ${\mathbf{e}_1}^{\otimes n+1}\otimes{\mathbf{e}_2}^{\otimes k+1}$ is a Lyndon word for all $n,k\geq 0$. Suppose we now expand $\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right)$ in terms of Lyndons words $\sum_{\text{Lyndon words }\mathbf{h}}\mathcal{P}_{\mathbf{h}}^*\circ\pi^{(n+k+2)}\left(\log S(\gamma)_{0,1}\right)\mathcal{P}_{\mathbf{h}}$, then for all $n,k\geq 0$,

$$\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(S(\gamma)_{0,1}\right)\quad=\quad\sum_{\text{Lyndon words }\mathbf{h}}\mathcal{P}_{\mathbf{h}}^*\circ\pi_N\left(\log S(\gamma)_{0,1}\right)\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(\mathcal{P}_{\mathbf{h}}\right).$$

By definition, $\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}(\mathcal{P}_{\mathbf{h}})$ will be none zero only if the word $\mathbf{h}$ contains $n+1$ letters $\mathbf{e}_1$ and $k+1$ letters $\mathbf{e}_2$. If $\mathbf{h}$ contains $n+1$ $\mathbf{e}_1$ and $k+1$ $\mathbf{e}_2$s, then by Lemma 16,

$$\mathcal{P}_{\mathbf{h}}=\mathbf{h}+\mathbb{Z}-\text{ linear combination of words greater than }\mathbf{h}. \tag{3.5}$$

However, $\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}$ is the smallest word amongst all words with $n+1$ $\mathbf{e}_1$s and $k+1$ $\mathbf{e}_2$s. Therefore, if $\mathbf{h}\neq\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}$, then the right hand side of (3.5) will only contain words strictly greater than $\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}$ and in particular will not contain the word $\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}$. Therefore, for all $n,k\geq 0$

$$\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}(\mathcal{P}_{\mathbf{h}})=0\text{ if }\mathbf{h}\neq\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}.$$

Therefore, for all $n,k\geq 0$,

$$\mathbf{e}_1^{*\otimes n+1}\otimes\mathbf{e}_2^{*\otimes k+1}\left(S(\gamma)_{0,1}\right)=\mathcal{P}^*_{\mathbf{e}_1^{\otimes n+1}\otimes\mathbf{e}_2^{\otimes k+1}}\circ\pi_N\left(\log S(\gamma)_{0,1}\right).$$

$\square$

We now prove Corollary 2.

Proof. (of Corollary 2)

In Example 15, we listed the Lyndon words of length less than or equal to 4. The corresponding Lyndon elements for the free Lie algebra generated by the alphabet $\{\mathbf{e}_1, \mathbf{e}_2\}$ is

$$\begin{aligned}\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]]], [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]], [\mathbf{e}_1, [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]], [\mathbf{e}_1, \mathbf{e}_2]\\ , [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], [[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2\end{aligned} \tag{3.6}$$

To prove Corollary 2, it is sufficient to express, for each of the above Lyndon elements f, the value $f\left(\log S(\gamma)_{0,1}\right)$ in terms of the winding number of γ.

As γ is a closed curve, $\mathbf{e}_i^*\left(\log\left(S(\gamma)_{0,1}\right)\right) = 0$ for $i = 1, 2$.

By Theorem 1,

$$\begin{aligned}
[\mathbf{e}_1, \mathbf{e}_2]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= \int_{\mathbb{R}^2} \eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y\\
[\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= \int_{\mathbb{R}^2} x\eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y\\
[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= -\int_{\mathbb{R}^2} y\eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y\\
[\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]]]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= \frac{1}{2}\int_{\mathbb{R}^2} x^2\eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y\\
[\mathbf{e}_1, [[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2]]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= -\int_{\mathbb{R}^2} xy\eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y\\
[[[\mathbf{e}_1, \mathbf{e}_2], \mathbf{e}_2], \mathbf{e}_2]^* \circ \pi_4\left(\log S(\gamma)_{0,1}\right) &= \frac{1}{2}\int_{\mathbb{R}^2} y^2\eta(\gamma - \gamma_0, (x, y))\, \mathrm{d}x\mathrm{d}y. \qquad (3.7)
\end{aligned}$$

□

3.3. **Sharpness of Corollary 2.**

The purpose of this section is to prove the following sharpness compliment to Corollary 2.

Proposition 20. *There exists two paths $\gamma, \tilde{\gamma}$ such that the winding number of γ and $\tilde{\gamma}$ around every point is equal, but the fifth term of their signature differs.*

Proof. Let $\mathbf{e}_i$ denote the path $t \to t\mathbf{e}_i$, $t \in [0, 1]$ and let

$$\gamma = \mathbf{e_1} \star \mathbf{e_2} \star -\mathbf{e_1} \star -\mathbf{e_2} \star -\mathbf{e_1} \star -\mathbf{e_2} \star \mathbf{e_1} \star \mathbf{e_2}$$

and

$$\tilde{\gamma} = -\mathbf{e_1} \star -\mathbf{e_2} \star \mathbf{e_1} \star \mathbf{e_2} \star \mathbf{e}_1 \star \mathbf{e}_2 \star -\mathbf{e}_1 \star -\mathbf{e}_2.$$

where $\star$ denote the concatenation operation on paths.

By Theorem 13 and the additivity of the winding number with respect to the concatenation product,

$$\eta(\gamma, (x, y)) = 1_{[0,1]\times[0,1]\cup[-1,0]\times[-1,0]}(x, y) = \eta\left(\tilde{\gamma}, (x, y)\right).$$

By a directly calculation, we see that the signature of $\mathbf{e}_i$ is

$$e^{\mathbf{e}_i}.$$

Therefore, by Chen's identity,

$$S(\gamma)_{0,1} \quad = \quad e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} \tag{3.8}$$

and

$$S(\tilde{\gamma})_{0,1} = e^{-\mathbf{e}_1} e^{-\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{\mathbf{e}_1} e^{\mathbf{e}_2} e^{-\mathbf{e}_1} e^{-\mathbf{e}_2}. \tag{3.9}$$

We claim that

$$\mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\gamma)_{0,1} \right) = 1$$

and

$$\mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\tilde{\gamma})_{0,1} \right) = -1.$$

Note that the word $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1$ is "square-free", i.e. none of the letter in the word is identical to the letter on its immediate left or right. This means the contribution to the value of both

$$\mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\gamma)_{0,1} \right)$$

and

$$\mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\tilde{\gamma})_{0,1} \right)$$

only comes from the first order term in exponentials in (3.8) and (3.9). For both, the contribution can only comes in one of the following five combinations:

Combination 1. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}$ exponentials.
Combination 2. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 7^{th}$ exponentials.
Combination 3. $1^{st}, 2^{nd}, 3^{rd}, 6^{th}, 7^{th}$ exponentials.
Combination 4. $1^{st}, 2^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.
Combination 5. $1^{st}, 4^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.

For $S(\gamma)_{0,1}$, the contributions from Combination 1 and Combination 5 is -1, while the contribution from Combination $2-4$ is 1. Therefore,

$$\begin{aligned} & \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\gamma)_{0,1} \right) \\ = \; & -1+1+1+1-1 \\ = \; & 1. \end{aligned}$$

For $S(\gamma')_{0,1}$, the contributions from Combination 1 and Combination 5 is 1, while the contribution from Combination $2-4$ is -1. Therefore,

$$\begin{aligned} & \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \otimes \mathbf{e}_2^* \otimes \mathbf{e}_1^* \left(S(\tilde{\gamma})_{0,1} \right) \\ = \; & 1-1-1-1+1 \\ = \; & -1. \end{aligned}$$

□

3.4. "tree-like" paths and winding number.

Proposition 21. *Let $1 \leq p < 2$. If a two dimensional path γ with finite p-variation has trivial signature then γ is closed and has winding number zero around all points (x,y) in $\mathbb{R}^2 \backslash \gamma[0,1]$.*

Proof. As the first term of the signature of γ is zero, we have

$$\int_0^1 \mathrm{d}\gamma = \gamma_1 - \gamma_0 = 0.$$

By Theorem 1,

$$\int_{\mathbb{R}^2} \frac{x^n y^k}{n!k!} \eta(\gamma - \gamma_0, (x,y))\, \mathrm{d}x\mathrm{d}y = 0$$

for all $n, k \geq 0$. Therefore,

$$\int_{\mathbb{R}^2} e^{\lambda_1 x + \lambda_2 y} \eta \left(\gamma - \gamma_0, (x, y)\right) \mathrm{d}x \mathrm{d}y = 0$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$. As the function $(x, y) \to \eta \left(\gamma - \gamma_0, (x, y)\right)$ lies in L^2 by (2.8), we have by the injectiveness of Fourier transform on L^2 that

$$\eta \left(\gamma, (x, y) + \gamma_0\right) = \eta \left(\gamma - \gamma_0, (x, y)\right) = 0$$

for all $(x, y) \in \mathbb{R}^2$ except a Lebesgue null set. As the function $(x, y) \to \eta \left(\gamma, (x, y) + \gamma_0\right)$ is locally constant on $\mathbb{R}^2 \backslash \gamma [0, 1]$, we have

$$\eta \left(\gamma - \gamma_0, (x, y)\right) = 0$$

for all $(x, y) \in \mathbb{R}^2 \backslash \gamma [0, 1]$. □

Remark 22. In [10], it was proved that the signature of a path with bounded total variation is trivial if and only if the path is "tree-like" (See Definition 1.2 in [10]). Therefore, Proposition 21 means that a planar tree-like path has zero winding around every point in the plane.

Remark 23. The converse of Lemma 21 is not true. Let γ and $\tilde{\gamma}$ be the paths defined in the proof of Proposition 20 and η be the concatenation of γ and the reversal of $\tilde{\gamma}$. Then by the additivity of winding number with respect to the concatenation product, η has zero winding number around every point. As the signature of γ and $\tilde{\gamma}$ are different, we have by Chen's identity that the signature of η is not $\mathbf{1}$. Therefore, η does not have trivial signature.

4. Uniqueness of signature

4.1. **Proof of Theorem 3.** Let $p \geq 1$. For elements γ and $\tilde{\gamma}$ in $\mathcal{V}^p \left([0, T_2], \mathbb{R}^d\right)$ and $\mathcal{V}^p \left([0, T_1], \mathbb{R}^d\right)$, define a concatenation product $\star : \mathcal{V}^p \left([0, T_2], \mathbb{R}^d\right) \times \mathcal{V}^p \left([0, T_1], \mathbb{R}^d\right) \to \mathcal{V}^p \left([0, T_1 + T_2], \mathbb{R}^d\right)$ by

$$\begin{aligned} \gamma \star \tilde{\gamma} (u) &:= \gamma (u), \quad u \in [0, T_1], \\ \gamma \star \tilde{\gamma} (u) &:= \tilde{\gamma} (u - T_1) + \gamma (T_1) - \tilde{\gamma} (0), \quad u \in [T_1, T_1 + T_2] \end{aligned}$$

Before proving our main result, we need just two more technical lemmas. The first one is a simple consquence of the Jordan curve theorem.

Lemma 24. *Let $p < 2$. Let γ and $\tilde{\gamma}$ be two simple curves with finite p-variation such that $\gamma_0 = \tilde{\gamma}_0$, $\gamma_1 = \tilde{\gamma}_1$ and $\eta (\tilde{\gamma} \star \overleftarrow{\gamma}, (x, y)) = 0$ for all $(x, y) \in \mathbb{R}^2 \backslash \left(\gamma [0, 1] \cup \tilde{\gamma} [0, 1]\right)$. Then $\gamma [0, 1] = \tilde{\gamma} [0, 1]$.*

Proof. Assume for contradiction that there exists a $t \in (0, 1)$ such that $\tilde{\gamma}_\sigma \notin \gamma [0, 1]$. Let

$$\begin{aligned} s &:= \inf \left\{\tau \leq \sigma : \tilde{\gamma} [\tau, \sigma] \cap \gamma [0, 1] = \emptyset\right\} \\ t &:= \sup \left\{\tau \geq \sigma : \tilde{\gamma} [\sigma, \tau] \cap \gamma [0, 1] = \emptyset\right\}. \end{aligned}$$

Then $\tilde{\gamma}_s, \tilde{\gamma}_t \in \gamma [0, 1]$ and $s < \sigma < t$. Let $u, v \in [0, 1]$ be such that $\gamma_u = \tilde{\gamma}_s$ and $\gamma_v = \tilde{\gamma}_t$. As γ and $\tilde{\gamma}$ are both simple, then either $\tilde{\gamma}|_{[s,t]} \star \gamma|_{[u,v]}$ or $\tilde{\gamma}|_{[s,t]} \star \overleftarrow{\gamma|_{[u,v]}}$ is a simple closed curve. This shows that there exists a simple curve ξ starting from $\tilde{\gamma}_s$ and ending at $\tilde{\gamma}_t$ such that $\tilde{\gamma}|_{[s,t]} \star \xi$ is a simple closed curve.

Suppose $u < v$. By Jordan curve theorem $\tilde{\gamma}_\sigma$ lies in the closure of both the interior and the exterior of $\tilde{\gamma}|_{[s,t]} \star \xi$. Therefore, for any $\varepsilon > 0$, the Enclidean ball centred at $\tilde{\gamma}_\sigma$ with radius ε contains a point x_ε in the interior of $\tilde{\gamma}|_{[s,t]} \star \xi$ and a point y_ε in the exterior of $\tilde{\gamma}|_{[s,t]} \star \xi$. Therefore,

$$\left|\eta\left(\tilde{\gamma}|_{[s,t]} \star \xi, x_\varepsilon\right) - \eta\left(\tilde{\gamma}|_{[s,t]} \star \xi, y_\varepsilon\right)\right| = 1. \tag{4.1}$$

By taking ε small, x_ε and y_ε will be in the same connected component of

$$\mathbb{R}^2 \backslash \left(\tilde{\gamma}[0,s] \cup \xi[0,1] \cup \tilde{\gamma}[t,1] \cup \gamma[0,1]\right).$$

Therefore,

$$\begin{aligned} & \eta\left(\tilde{\gamma}|_{[0,s]} \star \overleftarrow{\xi} \star \tilde{\gamma}|_{[t,1]} \star \gamma, x_\varepsilon\right) \\ = \; & \eta\left(\tilde{\gamma}|_{[0,s]} \star \overleftarrow{\xi} \star \tilde{\gamma}|_{[t,1]} \star \gamma, y_\varepsilon\right). \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2) and using the additvity of winding number we have

$$\left|\eta\left(\tilde{\gamma} \star \overleftarrow{\gamma}, x_\varepsilon\right) - \eta\left(\tilde{\gamma} \star \overleftarrow{\gamma}, y_\varepsilon\right)\right| = 1,$$

which is a contradiction. □

The second technical lemma states that the image of a simple curve determines the curve.

Lemma 25. *Let γ and $\tilde{\gamma}$ be simple curves such that $\gamma_0 = \tilde{\gamma}_0$ and $\gamma_1 = \tilde{\gamma}_1$. If $\gamma[0,1] = \tilde{\gamma}[0,1]$, then there exists a continuous strictly increasing function $r(t)$ such that*

$$\gamma_{r(t)} = \tilde{\gamma}_t$$

for all $t \in [0,1]$.

Proof. Let γ^{-1} denote the inverse of the function $t \to \gamma_t$, which exists as γ is a simple curve.

Define a function $r : [0,1] \to [0,1]$ by $r(t) = \gamma^{-1} \circ \tilde{\gamma}(t)$.

As both γ and $\tilde{\gamma}$ are injective continuous functions and $\gamma[0,1] = \tilde{\gamma}[0,1]$, thus r is a bijective continuous function from $[0,1]$ to $[0,1]$. Hence it is monotone.

But $\gamma_0 = \tilde{\gamma}_0 = -1$, $\gamma_1 = \tilde{\gamma}_1 = 1$, so $r(0) = 0$ and $r(1) = 1$. Hence r is an increasing function and the result follows. □

We now prove Theorem 3.

Proof. (of Theorem 3)The only if direction follows from the invariance of signature under translation and reparametrisation.

Let $\gamma, \tilde{\gamma}$ be simple curves such that $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$. Let $\hat{\gamma} = \tilde{\gamma} + \gamma_0 - \tilde{\gamma}_0$, and so $\hat{\gamma}_0 = \gamma_0$. By the translation invariance of signature, $S(\hat{\gamma})_{0,1} = S(\gamma)_{0,1}$. We want to show $\hat{\gamma}$ and γ are reparametrisation of each other.

By Chen's identity,

$$S\left(\hat{\gamma} \star \overleftarrow{\gamma}\right)_{0,1} = \mathbf{1}.$$

Since $\gamma, \hat{\gamma}$ are simple curves, we have by Proposition 21 that

$$\eta\left(\hat{\gamma} \star \overleftarrow{\gamma}, (x,y)\right) = 0$$

for all $(x,y) \in \mathbb{R}^2 \backslash \hat{\gamma} \star \overleftarrow{\gamma}[0,1]$.

Thereforem, by Lemma 24 and Lemma 25, $\hat{\gamma}$ is a reparametrisation of γ. □

5. Uniqueness of signature for Schramm-Loewner Evolution

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space. Let $(B_t : t \geq 0)$ be a one-dimensional standard Brownian Motion. Let $0 < \kappa$. Let $z \in \overline{\mathbb{H}} \backslash \{0\}$. For each $\omega \in \Omega$, consider the initial value problem:

$$\frac{\mathrm{d}g_t (z, \omega)}{\mathrm{d}t} = \frac{2}{g_t (z, \omega) - \sqrt{\kappa} B_t (\omega)} \quad g_0 (z) = z \tag{5.1}$$

We shall recall the following facts about g_t from [21].

(1) For each ω, a unique solution to this equation exists up to time $T_z > 0$, where T_z is the first time such that $g_t - \sqrt{\kappa} B_t \to 0$ as $t \to T_z$.
(2) Define
$$H_t = \{z \in \mathbb{H} : t < T_z\} \text{ and } K_t = \mathbb{H} \backslash H_t$$
Then H_t is open and simply connected.
(3) For each time $t > 0$, g_t defines a conformal map from H_t onto $\mathbb{H}$. In particular, g_t is invertible.
(4) Let $\hat{f}_t (z) := g_t^{-1} (z + \sqrt{\kappa} B_t)$. There exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$, the limit
$$\hat{\gamma} (t, \omega) := \lim_{z \to 0, z \in \mathbb{H}} \hat{f}_t (z)$$
exists and $t \to \hat{\gamma} (t)$ is continuous. The two dimensional stochastic process $(\hat{\gamma}_t : t \geq 0)$ is called the *Chordal SLE$_\kappa$ curve.*

The Loewner correspondence from a continuous path $t \to B_t (\omega)$ to $t \to \hat{\gamma} (\cdot, \omega)$ is in fact deterministic and one-to-one. Therefore, the measure on the Brownian paths induces, through this correspondence, a measure on paths in $\overline{\mathbb{H}}$ from 0 to ∞, which we shall call the Chordal SLE$_\kappa$ measure in $\mathbb{H}$.

Theorem 26. *Let $\kappa \leq 4$. Let $\mathbb{P}^{0,\infty}_{\kappa,\mathbb{H}}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$. Then with probability one, the following holds:*

Proposition 27. *1.([21],Theorem 7.1 and Theorem 6.1)$\gamma : [0, \infty) \to \overline{\mathbb{H}}$ satisfies $\gamma_0 = 0$ and $\liminf_{t\to\infty} |\hat{\gamma}_t| = \infty$.*
2.([21], Theorem 6.1)For $0 \leq \kappa \leq 4$, $t \to \hat{\gamma}_t$ is a simple curve.

The fact that $\lim_{t\to\infty} \hat{\gamma}_t = \infty$ *a.s.* means that the signature $S(\hat{\gamma})_{0,\infty}$ will not be defined. Therefore, we shall follow [27] and opt to study the Chordal SLE$_\kappa$ curve in the unit disc $\mathbb{D}$, from -1 to 1. The Chordal SLE_κ measure in domain $\mathbb{D}$ with marked points -1 and 1 is defined as follows:

Definition 28. For $\kappa > 0$. Let $\mathbb{P}^{0,\infty}_{\kappa,\mathbb{H}}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$, D be a simply connected subdomain of $\mathbb{C}$, $a, b \in \partial D$ and f be a conformal map from $\mathbb{H}$ to D, with $f(0) = a$ and $f(\infty) = b$. Then the Chordal SLE$_\kappa$ measure in D with marked points a and b is defined as the measure $\mathbb{P}^{0,\infty}_{\kappa,\mathbb{H}} \circ f^{-1}$.

Remark 29. Although there is a one dimensional family of conformal maps f such that f maps $\mathbb{H}$ to D, 0 to a and ∞ to b, the scale invariance of the Chordal SLE measure in $\mathbb{H}$ means that the measure $\mathbb{P}^{0,\infty}_{\kappa,\mathbb{H}} \circ f^{-1}$ is the same no matter which member f in this one dimensional family we use.

Theorem 30. *([27], Section 4.1) Let $0 < \kappa \leq 4$. Let $\mathbb{P}_{\kappa,\mathbb{D}}^{-1,1}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then with probability one, γ has finite p-variation for any $p > 1 + \frac{\kappa}{8}$.*

We now prove our almost sure uniqueness theorem concerning the signature of SLE curves.

Proof. (of Theorem 4)Let D be a smooth bounded Jordan domain and $a, b \in \partial D$. Let A be the set of curves γ such that

1. $\gamma(0) = a$, $\gamma(1) = b$.
2. γ has with finite $\frac{13}{8}$ variation.
3. γ is simple.

Let $\mathbb{P}_{\kappa,D}^{a,b}$ be the Chordal SLE$_\kappa$ measure in D with marked points a and b. Since a conformal map from $\mathbb{D}$ to a bounded Dini-smooth Jordan domain D has bounded derivative up to the boundary. Therefore, by Theorem 30, the SLE$_\kappa$ curves in any bounded Dini-smooth Jordan domain has finite p variation for any $p > 1 + \frac{\kappa}{8}$. Moreover, a conformal map from a Jordan domain D to a Jordan domain D' is continuous and injective on $\overline{D}$. Hence by Theorem 26 the SLE curve in a Jordan domain D is also a simple curve. Therefore, as $\frac{13}{8} > 1 + \frac{4}{8}$, $\mathbb{P}_{\kappa,D}^{a,b}(A^c) = 0$ for all $\kappa \leq 4$.

Let $\gamma, \tilde{\gamma} \in A$ be such that $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$, then by Theorem 3, γ and $\tilde{\gamma}$ are reparametrisations of each other. $\square$

6. Expected signature and n-point functions

6.1. **n-point functions from expected signature.** We will need the following immediate consequence of the shuffle product formula.

Lemma 31. *Let $(k_1, l_1), \ldots, (k_n, l_n) \in \mathbb{N}^2$. Then*

$$\Pi_{i=1}^n \mathbf{e}_1^{*\otimes k_i} \otimes \mathbf{e}_2^{*\otimes l_i} \left(S(\gamma)_{0,1} \right) = \mathbf{e}_1^{*\otimes k_1} \otimes \mathbf{e}_2^{*\otimes l_1} \sqcup \ldots \sqcup \mathbf{e}_1^{*\otimes k_n} \otimes \mathbf{e}_2^{*\otimes l_n} \left(S(\gamma)_{0,1} \right)$$

where the operation $\sqcup$ is the shuffle product operation defined in Proposition 7.

Proof. This follows from an iterated use of Proposition 7. $\square$

A well-known observable in the theory of SLE is the following sequence of n-point functions:

Definition 32. Let $0 < \kappa \leq 4$. Let D be a bounded Jordan domain and $a, b \in \partial D$. Let $\mathbb{P}_{\kappa,D}^{a,b}$ denote the chordal SLE$_\kappa$ measure on D with marked points a, b. Let $\Phi(\gamma)$ denote the concatenation of γ with the positively oriented arc from b to a. We shall define the n-point function associated with the probability measure $\mathbb{P}_{\kappa,D}^{a,b}$ to be:

$$\Gamma_n(x_1, y_1, .., x_n, y_n) = \mathbb{P}_{\kappa,D}^{a,b}\left[(x_1, y_1), \ldots (x_n, y_n) \in \mathrm{Int}\Phi(\cdot)\right].$$

The $n-$point functions for SLE$_\kappa$ curves were first studied by O. Schramm who calculated the 1-point function explicitly in terms of hypergeometric functions (see [22]). Although PDEs can be written down for the n-point functions, the analytic expressions for general n and κ are not known. The only exception is $n = 2$, $D = \mathbb{H}$ and $\kappa = \frac{8}{3}$, which was predicted in [24] and computed rigorously in [3].

Proof. (of Theorem 5)

Let A be as in the proof of Theorem 4.

Let $\gamma \in A$. Let $\Phi(\gamma)$ denote the concatenation of γ with the positively oriented arc in ∂D from b to a. As $\Phi(\gamma)$ is a simple closed curve, $\eta(\Phi(\gamma),(x,y)) = 1_{\mathrm{Int}\Phi(\gamma)}(x,y)$. Then by Lemma 19, we have for each $\gamma \in A$, for all $(n_1,k_1,\ldots,n_N,k_N) \in \mathbb{N}^{2N}$

$$\begin{aligned}&\Pi_{i=1}^{N}\mathbf{e}_1^{*\otimes(n_i+1)}\otimes\mathbf{e}_2^{*\otimes(k_i+1)}\left(S(\Phi(\gamma))_{0,1}\right)\\ =\ & C_{\mathbf{n}}\int_{\mathbb{R}^{2N}}\Pi_{i=1}^{N}x_i^{n_i}y_i^{k_i}1_{(\mathrm{Int}\Phi(\gamma))^N}\mathrm{d}x_1\mathrm{d}y_1\cdots\mathrm{d}x_N\mathrm{d}y_N\end{aligned}$$

where $(\mathrm{Int}\Phi(\gamma))^n := \mathrm{Int}\Phi(\gamma)\times\ldots\times\mathrm{Int}\Phi(\gamma)$ (n times) and

$$C_{\mathbf{n},\mathbf{k}} := \Pi_{i=1}^{N}\frac{(-1)^{k_i}}{n_i!k_i!}.$$

By Lemma 31, for all $(n_1,k_1,\ldots,n_N,k_N)\in\mathbb{N}^{2N}$,

$$\Pi_{i=1}^{N}\mathbf{e}_1^{*\otimes n_i}\otimes\mathbf{e}_2^{*\otimes k_i}\left(S(\Phi(\gamma))_{0,1}\right) = \mathbf{e}_1^{*\otimes n_1}\otimes\mathbf{e}_2^{*\otimes k_1}\sqcup\ldots\sqcup\mathbf{e}_1^{*\otimes n_N}\otimes\mathbf{e}_2^{*\otimes k_N}\left(S(\Phi(\gamma))_{0,1}\right).$$

By taking linear combinations, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}e^{\sum_{i=1}^{N}\lambda_i x_i+\mu_i y_i}\mathbb{E}\left[1_{D^N}\right]\mathrm{d}x_1\cdots\mathrm{d}y_N\\ =\ &\sum_{n_1,\ldots,n_N,k_1\ldots k_N\geq 0}\Pi_{i=1}^{N}(\lambda_i)^{n_i}(-\mu_i)^{k_i}\,\mathbf{e}_1^{*\otimes(n_1+1)}\otimes\mathbf{e}_2^{*\otimes(k_1+1)}\sqcup\ldots\\ &\ldots\sqcup\mathbf{e}_1^{*\otimes(n_N+1)}\otimes\mathbf{e}_2^{*\otimes(k_N+1)}\left(\mathbb{E}\left[S(\Phi(\gamma))_{0,1}\right]\right)\end{aligned}$$

The result then follows by noting $\mathbb{E}\left[1_{D^N}(\cdot)\right] = \Gamma_N(\cdot)$. □

As we may determine the signature of $\Phi(\gamma)$ from the signature of γ using Chen's identity, this formula gives a relationship between the expected signature of the Chordal SLE measure and the n-point functions.

6.2. **Expected signature from n-point functions.** We may ask whether it is possible to obtain the expected signature from the n-point functions. Unfortunately, here we can do no better than the deterministic case and are only able to obtain an explicit formula only up to the fourth term. To obtain a simpler formula, we choose to study the Chordal SLE$_\kappa$ measure on $\frac{1}{2}(1+\mathbb{D})$ so that all paths start from 0.

Proposition 33. *Let $0<\kappa\leq 4$. Let γ denote the Chordal SLE$_\kappa$ curve from 0 to 1 in $\frac{1}{2}(1+\mathbb{D})$. Let $\Phi(\gamma)$ denote the concatenation of γ with the upper semi-circle of the unit disc $\frac{1}{2}(1+\mathbb{D})$, oriented in the anti-clockwise direction. Then the level-4 truncated expected signature of $\Phi(\gamma)$ is*

$$\begin{aligned}&1+\textstyle\int_{\mathbb{D}}\left([\mathbf{e}_1,\mathbf{e}_2]+[\mathbf{x}_1,[\mathbf{e}_1,\mathbf{e}_2]]+\frac{1}{2}[\mathbf{x}_1,[\mathbf{x}_1,[\mathbf{e}_1,\mathbf{e}_2]]]\right)\Gamma_1((x_1,y_1))\,\mathrm{d}x_1\mathrm{d}y_1\\ &+\tfrac{1}{2}\textstyle\int_{\mathbb{R}^4}[\mathbf{e}_1,\mathbf{e}_2]\otimes[\mathbf{e}_1,\mathbf{e}_2]\,\Gamma_2((x_1,y_1),(x_2,y_2))\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\end{aligned}\tag{6.1}$$

where $\mathbf{x}_1 = x_1\mathbf{e}_1+y_1\mathbf{e}_2$ and $\mathbf{x}_2 = x_2\mathbf{e}_1+y_2\mathbf{e}_2$, and Γ_n is the n-point function for the Chordal SLE$_\kappa$ measure.

Proof. Let A be the set defined in the proof Theorem 4.

Let $\gamma \in A$. As $\Phi(\gamma)$ is closed, $\mathbf{e}_1^*\left(\log S\left(\Phi(\gamma)_{0,1}\right)\right) = \mathbf{e}_2^*\left(\log S\left(\Phi(\gamma)_{0,1}\right)\right) = 0$. Hence by (3.7),

$$
\begin{aligned}
\pi_4\left(\log S\left(\Phi(\gamma)_{0,1}\right)\right) &= \int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} x\, [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} y\, [\mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} \frac{x^2}{2}\, [\mathbf{e}_1, [\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} xy\, [\mathbf{e}_1, [\mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2]]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} \frac{y^2}{2}\, [\mathbf{e}_2, [\mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2]]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2} \Big([\mathbf{e}_1, \mathbf{e}_2] + [x\mathbf{e}_1 + y\mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2]] \\
&\quad + \frac{1}{2}\, [x\mathbf{e}_1 + y\mathbf{e}_2, [x\mathbf{e}_1 + y\mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2]]] \Big)\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

By taking the exponential and writing $x\mathbf{e}_1 + y\mathbf{e}_2$ as $\mathbf{x}$,

$$
\begin{aligned}
\pi_4\left(S\left(\Phi(\gamma)_{0,1}\right)\right) &= 1 + \int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} [\mathbf{x}, [\mathbf{e}_1, \mathbf{e}_2]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} [\mathbf{x}, [\mathbf{x}, [\mathbf{e}_1, \mathbf{e}_2]]]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{6.2}
$$

Note that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^4} [\mathbf{e}_1, \mathbf{e}_2] \otimes [\mathbf{e}_1, \mathbf{e}_2]\, 1_{\mathrm{Int}\Phi(\gamma)\times\mathrm{Int}\Phi(\gamma)}(x_1, y_1, x_2, y_2)\, \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2.
\end{aligned}
$$

The proof is completed by taking expectation. □

Before we calculate the fourth term of the expected signature of SLE curves, we need the expected signature of a semi-circle with radius $\frac{1}{2}$.

Lemma 34. *The first four terms in the signature of a semi-circle is*

Proof. By translation invariance of signature, we may calculate instead the signature of the semi-circle $\phi \star \psi$ where $\phi(t) := \frac{1}{2}(\cos t, \sin t)$, $t \in [0, \pi]$ and $\psi(t) := (1 - t, 0)$ $t \in [0, 1]$. By exactly the same computation required to obtain (6.2),we have

$$\begin{aligned}\pi_4\left(S\left(\phi\star\psi\right)_{0,1}\right) &= 1-\int_{\mathbb{R}^2}\left[\mathbf{e}_1,\mathbf{e}_2\right]1_{\mathrm{Int}\phi\star\psi}\left(x,y\right)\mathrm{d}x\mathrm{d}y\\ &\quad-\int_{\mathbb{R}^2}\left[\mathbf{x},\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]1_{\mathrm{Int}\phi\star\psi}\left(x,y\right)\mathrm{d}x\mathrm{d}y\\ &\quad-\int_{\mathbb{R}^2}\left[\mathbf{x},\left[\mathbf{x},\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]1_{\mathrm{Int}\phi\star\psi}\left(x,y\right)\mathrm{d}x\mathrm{d}y\\ &\quad+\int_{\mathbb{R}^2}\left[\mathbf{e}_1,\mathbf{e}_2\right]1_{\mathrm{Int}\phi\star\psi}\left(x,y\right)\mathrm{d}x\mathrm{d}y\otimes\int_{\mathbb{R}^2}\left[\mathbf{e}_1,\mathbf{e}_2\right]1_{\phi\star\psi}\left(x,y\right)\mathrm{d}x\mathrm{d}y \qquad (6.3)\end{aligned}$$

where the negative sign is due to that $\phi\star\psi$ has negative orientation.

By changing to polar coordinate, (6.3) can be calculated to be

$$\begin{aligned}&1-\frac{\pi}{4}\left[\mathbf{e}_1,\mathbf{e}_2\right]-\frac{1}{12}\left[\mathbf{e}_1.\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]-\frac{\pi}{128}\left[\mathbf{e}_1,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\\ &-\frac{\pi}{128}\left[\mathbf{e}_2,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]+\frac{\pi^2}{16}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right].\end{aligned}$$

∎

Theorem 35. *Let $0<\kappa\le 4$. The fourth term in the expected signature of SLE$_\kappa$ curve in $\frac{1}{2}\left(1+\mathbb{D}\right)$ is*

Proof. By Chen's identity,

$$\begin{aligned}&\left(1+\int_{\mathbb{D}}\left(\left[\mathbf{e}_1,\mathbf{e}_2\right]+\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]+\frac{1}{2}\left[\mathbf{x}_1,\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\right)\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\right.\\ &\left.+\frac{1}{2}\int_{\mathbb{R}^4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\right)\\ \otimes\;&\left(1-\frac{\pi}{4}\left[\mathbf{e}_1,\mathbf{e}_2\right]-\frac{1}{12}\left[\mathbf{e}_1.\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]-\frac{\pi}{128}\left[\mathbf{e}_1,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\right.\\ &\left.-\frac{\pi}{128}\left[\mathbf{e}_2,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]+\frac{\pi^2}{16}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\right)\\ \otimes\;&e^{\mathbf{e}_1}\\ =\;&\int_{\mathbb{D}}\frac{1}{2}\left[\mathbf{x}_1,\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1+\frac{1}{2}\int_{\mathbb{R}^4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\\ &+\int_{\mathbb{D}}\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\mathbf{e}_1\\ &+\int_{\mathbb{D}}\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\frac{\mathbf{e}_1^{\otimes 2}}{2}-\frac{\pi}{4}\int_{\mathbb{D}}\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\\ &-\frac{\pi}{128}\left[\mathbf{e}_1,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]-\frac{\pi}{128}\left[\mathbf{e}_2,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]+\frac{\pi^2}{16}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\\ &-\frac{1}{12}\left[\mathbf{e}_1.\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\otimes\mathbf{e}_1-\frac{\pi}{4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\frac{\mathbf{e}_1^{\otimes 2}}{2}+\frac{\mathbf{e}_1^{\otimes 4}}{4!}\end{aligned}$$

However, by the symmetry of SLE curve, $\mathbf{e}_{i_1}^*\otimes\mathbf{e}_{i_2}^*\otimes\mathbf{e}_{i_3}^*\otimes\mathbf{e}_{i_4}^*\left(\mathbb{E}\left(S\left(\gamma\right)_{0,1}\right)\right)=0$ if $\left(i_1,i_2,i_3,i_4\right)$ contains an odd number of 2s. Therefore, we only need to look at terms with an even number of 2s, namely,

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{1}{2}\left[\mathbf{x}_1,\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1+\frac{1}{2}\int_{\mathbb{R}^4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\\
&+\int_{\mathbb{D}}\left[\mathbf{x}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\mathbf{e}_1\\
&+\int_{\mathbb{D}}\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\frac{\mathbf{e}_1^{\otimes 2}}{2}-\frac{\pi}{4}\int_{\mathbb{D}}\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\\
&-\frac{\pi}{128}\left[\mathbf{e}_1,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]-\frac{\pi}{128}\left[\mathbf{e}_2,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]+\frac{\pi^2}{16}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\\
&-\frac{1}{12}\left[\mathbf{e}_1.\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\otimes\mathbf{e}_1-\frac{\pi}{4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\frac{\mathbf{e}_1^{\otimes 2}}{2}+\frac{\mathbf{e}_1^{\otimes 4}}{4!}\\
=\;&\frac{1}{2}\int_{\mathbb{D}}x_1y_1\left(\left[\mathbf{e}_1,\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]+\left[\mathbf{e}_2,\left[\mathbf{e}_1,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\right]\right)\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\\
&+\frac{1}{2}\int_{\mathbb{R}^4}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_2\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\\
&+\int_{\mathbb{D}}y\left[\mathbf{e}_2,\left[\mathbf{e}_1,\mathbf{e}_2\right]\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\mathbf{e}_1\\
&-\frac{\pi}{4}\int_{\mathbb{D}}\left[\mathbf{e}_1,\mathbf{e}_2\right]\Gamma_1\left(x_1,y_1\right)\mathrm{d}x_1\mathrm{d}y_1\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]\\
&+\frac{\pi^2}{16}\left[\mathbf{e}_1,\mathbf{e}_2\right]\otimes\left[\mathbf{e}_1,\mathbf{e}_2\right]+\frac{\mathbf{e}_1^{\otimes 4}}{4!}.
\end{aligned}$$

□

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